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Finite lattice Bethe ansatz systems and the Heun equation

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Abstract

We study the Pöschl–Teller equation in complex domain and deduce infinite families of TQ and Bethe ansatz equations, classified by four integers. In all these models the form of T is very simple, while Q can be explicitly written in terms of the Heun function. At particular values there is an interesting interpretation in terms of finite lattice spin-$\frac{L-2}{2}$ XXZ quantum chain with $\Delta = \cos\frac{\pi}{L}$ (for free–free boundary conditions), or $\Delta = -\cos\frac{\pi}{L}$ (for periodic boundary conditions). This result generalizes the findings of Fridkin, Stroganov and Zagier. We also discuss the continuous (field theory) limit of these systems in view of the so-called ODE/IM correspondence.

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1. Introduction

Some years ago, an unexpected connection was found between certain (0+1)-dimensional quantum-mechanical problems and (1+1)-dimensional conformal field theories [1–8]. The simplest example involves the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x,E) + (x^\alpha - E)\psi(x,E) = 0 \tag{1.1}$$

and the fact that (1.1) has unique solution $y(x,E)$, entire in x and E, which decays along the positive real axis as $x \to \infty$. The function $y(x,E)$ can be shown [5] to satisfy a Stokes relation

$$T(E)y(x,E) = \omega^{-1/2}y(\omega x, \omega^{-2}E) + \omega^{1/2}y(\omega^{-1}x, \omega^2 E) \tag{1.2}$$

with $\omega = \exp(i2\pi/(\alpha+2))$. This is strikingly similar to the TQ relation, a functional equation which was introduced in the context of the six-vertex model by Baxter (cf [9]).

The correspondence is actually much more precise: the Stokes multiplier [10] $T(E)$ and the spectral determinants [11] $Q^{+}(E) = y(x, E)|_{x=0}$ and $Q^{-}(E) = y'(x, E)|_{x=0}$ are equal to the ground-state eigenvalues of the transfer matrix and of the Q operators, respectively, of the six-vertex or, equivalently, of the spin-$\frac{1}{2}$ XXZ quantum chain, in suitable continuum (field theory) limits. The continuum limit of the six-vertex model is related to conformal theory at $c = 1$, and the description of such theories in a framework similar to Baxter's lattice set-up was addressed in an important series of works by Bazhanov *et al* [12–14]. The correspondence also applies, directly, in this context. Since the initial observation of [1], many mathematical aspects of the correspondence have been clarified [2, 3, 5], but the physical reasons for its existence remain mysterious. In addition, up to now, the correspondence has been limited to the ground-state energy of the conformal field theory and it is unclear whether it can be extended to massive theories, to excited states[4], or to finite lattice systems[5].

The purpose of this paper is to report some progress on the last of these questions. The key difference in finite lattice problems is the appearance of an extra function $\Phi(E)$ in the *TQ* relation:

$$T(E)Q(E) = b^{-1}\Phi(\omega E)Q(\omega^{-2}E) + b\Phi(\omega^{-1}E)Q(\omega^{2}E). \tag{1.3}$$

(b is a pure phase.) This function is fixed by the problem under consideration. In particular, it encodes the number of lattice sites, and tends to 1 when a suitable continuum limit is taken.

We start with the fact that at the points $\alpha = 2$ and $\alpha = 1$ the solutions of (1.1) can be given explicitly, in terms of hypergeometric functions. In the search for finite-lattice generalizations of the correspondence it seems natural to begin with these two models. At the linear point, $\alpha = 1$, there is the additional advantage that the rôles of E and x are interchangeable: setting $z = x - E$, (1.1) becomes the standard Airy equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z) + z\psi(z) = 0 \tag{1.4}$$

and therefore [1, 4]

$$Q^{+}(E)|_{\alpha=1} = \mathrm{Ai}(-E) \qquad \text{and} \qquad Q^{-}(E)|_{\alpha=1} = -\mathrm{Ai}'(-E) \tag{1.5}$$

which means that the functions $Q^{\pm}(E)$ themselves satisfy differential equations.

The ideal situation would be that the finite-lattice version of the problem would share this (E, x)-democracy. Surprisingly, this was precisely the discovery made by Fridkin *et al* in [19]: though they did not make a connection with the earlier results of [1–5], they empirically discovered that the Q function for the spin-$\frac{1}{2}$ XXZ quantum chain with free–free boundary conditions and $\Delta = 1/2$ was related to a specialization of the Pöschl–Teller equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{9n(2n+1)}{2\cosh^2 3s/2}\right)\chi(s) = -\chi(s). \tag{1.6}$$

In a subsequent paper [20], the same spin chain, but with $\Delta = -1/2$ and with periodic boundary conditions, was also related to the Pöschl–Teller equation with a different eigenvalue.

In this paper we show that the most general Pöschl–Teller equation (given in equation (2.1)) contains infinite families of finite *TQ* and Bethe ansatz equations, selected by fixing the four parameters (M, N, L, m) to positive integer values. At the particular values $(0, N, L, 0)$ and $(0, N, L, 1)$ there is a straightforward and interesting interpretation in terms of finite lattice spin-$\frac{L-2}{2}$ XXZ quantum chains.

4 While we were finishing writing this paper the preprint [15] appeared. In this very interesting paper, Schrödinger equations corresponding to excited states are proposed.

5 An extension of the finite lattice Baxter *TQ* relation with extra coordinates like parameters has been recently introduced by Weston and Rossi in [16] for q generic (see also [17, 18] for q a root of unity). The relationship between these results and the ODE/IM correspondence still needs to be clarified.

The plan of the paper is as follows. The relevant equation and its analytic properties are discussed in section 2. The *TQ* relation is derived in section 3 and the relationship with the quantum spin chains is discussed in section 4. In section 5 some numerical results are reported and in section 6 the continuous limit of the equation is studied in view of the ODE/IM correspondence. Finally section 7 contains our conclusions. There are two appendices: in appendix A the solution in terms of the Heun function is derived, while the locations of the trivial zeros of the solution are discussed in appendix B.

2. The differential equation

We consider the generalized Pöschl–Teller equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{N(N+1)}{\cosh^2 s} + \frac{M(M+1)}{\sinh^2 s}\right)\chi(s) = -\sigma^2\chi(s). \tag{2.1}$$

As explained in appendix A, the differential equation (2.1) can be mapped into the Heun equation, allowing any solution of (2.1) to be written in terms of the Heun function H. This is one of the reasons why the Pöschl–Teller equation has historically played an important rôle in the quantum-mechanical modelling of two-body problems with short-range interactions. In these applications the wavefunction is usually defined on the real axis, and the physical requirement of square integrability constrains σ to integer values, allowing the solution to be written in terms of the more standard ${}_2F_1$ hypergeometric function.

In this paper we shall instead consider equation (2.1) on the whole complex plane, and one of the requirements placed on its solutions will be meromorphicity. The demand that $\chi(s)$ be single valued around the singularities of $\cosh^{-2} s$ and $\sinh^{-2} s$ immediately restricts the parameters N and M to integer values. However, in the following we shall impose further conditions, which emerge as follows.

Introduce a new variable $x^L = -\exp(2s)$, and set

$$m = \sigma L - 1 \qquad \psi_{M,N,m}(x) = x^{(m+1)/2}\chi(\ln(\sqrt{-x^L})). \tag{2.2}$$

Then $\psi_{M,N,m}(x)$ is solution of

$$\left(x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - mx\frac{\mathrm{d}}{\mathrm{d}x} - \frac{L^2N(N+1)x^L}{(x^L-1)^2} + \frac{L^2M(M+1)x^L}{(x^L+1)^2}\right)\psi_{M,N,m}(x) = 0. \tag{2.3}$$

The requirement that $\psi_{M,N,m}(x)$ be single valued on the whole complex plane leads to the quantization of the four parameters N, M, L, m to integer values. To see the quantization of m, note that the points $x = 0$ and $x = \infty$ are, in general, singular points of (2.3), and in their vicinity solutions behave as

$$\psi_{M,N,m}(x) \sim \alpha + \beta x^{m+1} + \cdots \tag{2.4}$$

and therefore single valuedness constrains $m \in \mathbb{Z}$. The case $m \geqslant 0$ is already very rich in structure, and so we shall restrict ourselves to this case. Without any further loss of generality, we conventionally set $L, N, M \geqslant 0$. All this was to allow the single valuedness and hence meromorphicity of the solutions to (2.3). To single out one particular solution, to play the rôle of the function y in (1.1), we shall impose the boundary condition

$$\psi_{M,N,m}(x)|_{x\sim 1} \sim (1-x)^{N+1}. \tag{2.5}$$

This condition is natural, in that (2.3) has regular singularities at

$$(x^L \mp 1)|_{x=x_{i,\pm}} = 0. \tag{2.6}$$

Near these points a generic solution behaves as

$$\psi_{M,N.m}(x) \sim c(x - x_{i-})^{-N} \qquad (x^L \sim 1) \tag{2.7}$$

$$\psi_{M,N,m}(x) \sim \tilde{c}(x - x_{i+})^{-M} \qquad (x^L \sim -1). \tag{2.8}$$

If we impose $c = 0$ then, exceptionally,

$$\psi_{M,N,m}(x) \sim \mathrm{d}(x - x_{i-})^{N+1} \tag{2.9}$$

which up to the (arbitrary) normalization is exactly the condition (2.5). The choice to impose boundary conditions near regular singularities in this way might seem to be unmotivated at this stage, but it will be crucial in making a connection with (1.1) in the scaling limit. We shall return to this point in section 6.

Note that equation (2.3) is invariant under the transformation

$$(x, M, N) \to (x\omega^{1/2}, N, M) \tag{2.10}$$

where $\omega = \exp(2\mathrm{i}\pi/L)$, and consequently also under

$$(x, M, N) \to (x\omega, M, N). \tag{2.11}$$

These symmetries force further zeros in $\psi_{M,N,m}(x)$. Being images of $x = 1$ under certain rotations, they are located on the unit circle, and are, in some sense, trivial. They will, however, contribute non-trivially to the Bethe ansatz equations which fix the nontrivial zeros—see, for example, (3.11). The determination of the locations of the trivial zeros is simple but technical, and we relegate it to appendix B.

3. The connection formula and the TQ relation

We shall now formulate the problem in a set-up similar to that used in presence of Stokes sectors [10]. Set $g(x) = \psi_{M,N,m}(x)$ and define

$$g_k(x) = g(\omega^k x). \tag{3.1}$$

Then the symmetry (2.11) ensures that $g_1(x)$ and $g_{-1}(x)$ are also solutions of equation (2.3). Near $x = 1$ they behave as

$$g_1(x) \sim c_+(x - 1)^{-N} \qquad g_{-1}(x) \sim c_-(x - 1)^{-N} \tag{3.2}$$

and so the pair of functions $\{g_0(x), g_1(x)\}$ is (apart from the particular values of $m = L - 1 \bmod L$) a basis of solutions. Expanding g_{-1} in this basis and rearranging,

$$W[-1, 1]g_0(x) = W[-1, 0]g_1(x) + W[0, 1]g_{-1}(x) \tag{3.3}$$

where the Wronskian $W[i, j]$ is

$$W[i, j] = g_i(x)g'_j(x) - g'_i(x)g_j(x). \tag{3.4}$$

Because of the term $-mx\frac{\mathrm{d}g(x)}{\mathrm{d}x}$ in (2.3), one can deduce that the Wronskian between any pair of solutions has the form

$$W[g, f] = cst\, x^m \tag{3.5}$$

and one can factorize x^m out of (3.3). We can now use the large x asymptotic (2.4)

$$g(x) \sim a + bx^{m+1} + \cdots \tag{3.6}$$

to find the exact expression for $W[-1, 0]$ and $W[0, 1]$. For $a, b \neq 0$ the result is

$$W[-1, 0] = 2\mathrm{i}ab(m+1)\omega^{(m+1)/2} \sin\left(\tfrac{m+1}{L}\pi\right)x^m \tag{3.7}$$

$$W[0, 1] = 2\mathrm{i}ab(m+1)\omega^{-(m+1)/2} \sin\left(\tfrac{m+1}{L}\pi\right)x^m \tag{3.8}$$

$$W[-1, 1] = 4\mathrm{i}ab(m+1) \sin\left(\tfrac{m+1}{L}\pi\right) \cos\left(\tfrac{m+1}{L}\pi\right)x^m, \tag{3.9}$$

and (3.3) becomes

$$2\cos\left(\tfrac{m+1}{L}\pi\right)g(x) = \omega^{\frac{m+1}{2}} g(\omega^{-1}x) + \omega^{-\frac{m+1}{2}} g(\omega x). \tag{3.10}$$

This equation is almost identical to Baxter's *TQ* relation, save for the fact that $Q(x)$ is, by definition, entire while $g(x) = \psi_{M,N,m}(x)$ is not. This can be simply overcome by introducing a new function $q(x)$ defined as

$$(x^L - 1)^N (x^L + 1)^M g(x) = \prod_{j=0}^{\ell-1} (x - (\omega')^j)^{2N+1} \prod_{i=1}^{N_k} \left(x - (\omega)^{\frac{2k_i+1}{2}}\right)^{2M+1} q(x). \tag{3.11}$$

In the above we have used the knowledge of the trivial zeros and poles discussed in appendix B. Note that ℓ, ω', k_i and N_k are, respectively, defined in (B.5), (B.6), (B.7) and at the end of appendix B.

By a consideration of the possible singularities and trivial zeros in the previous section and in appendix B, we immediately deduce the following factorized form for $q(x)$:

$$q(x) = \prod_{i=1}^{K} \left(1 - \frac{x}{x_j}\right) \tag{3.12}$$

with

$$x_j = 1/x_{K+1-j}. \tag{3.13}$$

The function $q(x)$ also satisfies a *TQ*-type relation, and it is entire.

The number of nontrivial zeros, K, of $q(x)$ is easily evaluated by noting that $\psi_{M,N,m}(x)$ is a meromorphic function of x. Then, the asymptotic behaviour (3.6) indicates that $\mathrm{Max}(m+1, 0)$ should be equal to the total number of zeros minus the total number of poles existing at finite x. For $m + 1 > 0$, this leads to

$$K = (m+1) + N(L-\ell) + M(L-N_k) - \ell(N+1) - N_k(M+1). \tag{3.14}$$

The set of numbers $\{x_j\}$ constitutes the nontrivial zeros of the wavefunction. They are fixed by the Bethe ansatz equations. To match the standard notation, we change variables $x \to -x$ and $E_j = -x_j$, and also set $x = \exp(u)$, $E_j = \exp(u_j)$. In these new variables the connection (*TQ*) formula (3.10) becomes

$$\tau(u)Q(u) = \phi(u - 2\mathrm{i}\eta)Q(u - 2\mathrm{i}\eta) + \phi(u + 2\mathrm{i}\eta)Q(u + 2\mathrm{i}\eta) \tag{3.15}$$

where we defined

$$Q(u) = \prod_{j=1}^{K} \sinh\left(\frac{u - u_j}{2}\right) \tag{3.16}$$

$$\phi(u) = \prod_{j=0}^{\ell-1} \cosh^{2N+1}\left(\frac{u}{2} - i\eta' j\right) \prod_{i=1}^{N_k} \cosh^{2M+1}\left(\frac{u}{2} - i\frac{2k_i+1}{2}\eta\right) \tag{3.17}$$

$$\tau(u) = (-1)^{N+M} 2\cos\left(\tfrac{m+1}{L}\pi\right)\phi(u) \tag{3.18}$$

with $\eta = \pi/L$ and $\eta' = \pi/\ell$.

From its explicit form, $\tau(u)$ should be pole-free while the formal solution of the above algebraic equation seems to possess poles at the zeros of $Q(u)$. Thus, the residue at these points must be vanishing. This is exactly the same reasoning which leads to the Bethe ansatz equation in integrable systems. A suitably-chosen solution to the resulting Bethe ansatz equation characterizes $Q(u)$, and consequently $q(x)$, and exhibits several interesting patterns depending on the choice of parameters. Before presenting examples, however, we shall discuss the connection of our findings to quantum magnets.

4. Model identification

Consider a one-dimensional quantum system in which quantum spins of magnitude S are assigned to each site of a length $\mathcal{N}_S$ chain. They interact via spins of magnitude 1/2 living on bonds. The strength of the interaction is characterized by $\triangle = \cos\lambda$. Assume further either periodic boundary conditions (p.b.c), or free–free (f–f) boundary conditions with $U_q(sl_2)$ invariant interaction and $q = \exp \mathrm{i}\lambda$. For example, the Hamiltonian for $S = \frac{1}{2}$, and periodic boundary conditions, is

$$\mathcal{H} = \sum_{n=1}^{\mathcal{N}_{1/2}} \left(\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\Delta}{2} \sigma_n^z \sigma_{n+1}^z \right).$$

The transfer matrix $T_S(u)$ is given by either a single (p.b.c) or doubled (f–f) form via Skylanin's construction [21]. The auxiliary space has spin $\frac{1}{2}$, and the quantum space is given by the $\mathcal{N}_S$-fold tensor product of a spin S space. Then the following TQ relation holds:

$$\begin{aligned}
&T_S^{(r)}(u) Q_S(u) = \phi_S(u - 2\mathrm{i}S\lambda) Q_S(u + 2\mathrm{i}\lambda) + \phi_S(u + 2\mathrm{i}S\lambda) Q_S(u - 2\mathrm{i}\lambda)\\
&\qquad Q_S(u) = \begin{cases} \prod_j \sinh \frac{u - v_j}{2} & \text{(p.b.c)} \\ \prod_j \sinh \frac{u - v_j}{2} \sinh \frac{u + v_j}{2} & \text{(f–f)} \end{cases}\\
&\qquad \phi_S(u) := \begin{cases} \prod_{\alpha=1}^{\mathcal{N}_S} \sinh \frac{(u - \omega_\alpha)}{2} & \text{(p.b.c)} \\ \sinh(u) \prod_{\alpha=1}^{\mathcal{N}_S} \sinh \frac{(u - \omega_\alpha)}{2} \sinh \frac{(u + \omega_\alpha)}{2} & \text{(f–f)} \end{cases}
\end{aligned} \tag{4.1}$$

where ω_α stands for the inhomogeneity and $T_S^{(r)}(u)$ stands for the renormalized transfer matrix:

$$T_S^{(r)}(u) = \begin{cases} T_S(u) & \text{(p.b.c)} \\ \sinh u \; T_S(u) & \text{(f–f).} \end{cases}$$

For periodic boundary conditions, the above relation can be shown directly, while for free–free boundaries, it generalizes established results for $S = \frac{1}{2}$ and 1 [22, 23].

The similarity between (4.1) and our connection formula (3.15–3.18) is clear. To check the precise correspondence, we now examine some simple examples, taking $M = 0$ and $m = 0$ or 1. For $m = 1$, we additionally impose that L be odd, so that in all cases $\ell = 1$. Then ϕ in (3.17) simplifies considerably:

$$\phi(u) = \begin{cases} \cosh^{2N+1}\left(\frac{u}{2}\right) & (N_k = 0) \\ \frac{1}{2}\sinh(u) \cosh^{2N}\left(\frac{u}{2}\right) & (N_k = 1). \end{cases}$$

Noting also the property

$$\cosh\left(\frac{u}{2} \pm \mathrm{i}\frac{\pi}{L}\right) = \pm \mathrm{i} \sinh\left(\frac{u}{2} \mp \mathrm{i}\frac{L-2}{2L}\pi\right) = \pm \mathrm{i} \sinh\left(\frac{u}{2} \mp \mathrm{i}(L-2)\frac{\eta}{2}\right) \tag{4.2}$$

for $N_k = 1$ it is immediately seen that the connection rule (3.15–3.18) coincides with (4.1) for the spin $\frac{L-2}{2}$ chain with f–f boundaries, an even number of sites $\mathcal{N}_S = 2N$, and with parameters $\eta = \lambda = \frac{\pi}{L}$, $u_j = v_j$ and $\omega_\alpha = 0$. For the match to be complete, the function $\tau(u)$ should be related to an eigenvalue $T_S^{(r)}(u)$ of the transfer matrix as $2(-1)^{N+1}\tau(u) = T_S^{(r)}(u)$.

Similarly, for $N_k = 0$ and L odd, the connection rule coincides with (4.1) for the spin $\frac{L-2}{2}$ chain, but with p.b.c and an odd number of sites $\mathcal{N}_S = 2N + 1$. The parameters must be identified as $\lambda = \pi - \eta = \frac{(L-1)\pi}{L}$, $u_j = v_j + \pi\mathrm{i}$, $(-1)^{\frac{L}{2}+m-1}\tau(u) = T_S^{(r)}(u + \pi\mathrm{i})$ and $\omega_\alpha = 0$.

Specializing to $L = 3$, the above results recover the findings of [19, 20]. The coincidence between the ODE and the spin chain was checked numerically for $N_k = 0$, $L = 3, 5$, with $N = 1, 2, 3$. We adopted a 'brute force' diagonalization of the transfer matrices associated with the spin systems, and then verified that the resultant spectra contain eigenvalues of the form $T_S^{(r)}(u) = 2(-1)^{N+1}\tau(u)$. These correspond to the particular solutions of the Bethe ansatz equations (5.1) which will be associated with the ODE in the next section. The eigenvalues are not particular members of the spectra: they are neither the largest in magnitude nor the smallest. However, the same BAE patterns do play a distinguished rôle in a particular (isotropic) fused model. See the discussion in section 7. The remarkable simplicity of the expression for $T_S(u)$ comes from the elementary expression for $\tau(u)$ in (3.18); it reflects the special nature of the points we are examining even before the scaling limit is taken.

When $N_k = 0$ and L is even, our connection rule differs from the periodic boundary condition case of (4.1) by a sign. This suggests the need for a different choice of boundary conditions for the spin model. We leave this for future work, as well as the identification of the connection rule and the TQ relation in higher spin chains, with general choice of N, M, L, m, where the inhomogeneities ω_α should be chosen properly.

We make one further, more general, remark in concluding this section. There are some ambiguities in the choice of functions in the lattice model and the ODE : in particular, equation (3.10) is invariant if $\tau(u)$ and $\phi(u)$ are multiplied by a common arbitrary entire function of x^L.[6] Although the choice we have made above appears to be the most natural, and is supported by our numerical results, we cannot exclude the possibility of the extra factor being relevant in a more general situation. We hope to resolve this issue in a future publication.

5. The Bethe ansatz equations and string-like solutions

From equation (3.15), and the reasoning given after that equation, the zeros $\{u_j\}$ of $Q(u)$ satisfy the following Bethe Ansatz equations (BAE)[7]

$$\frac{\phi(u_j - 2\mathrm{i}\eta)Q(u_j - 2\mathrm{i}\eta)}{\phi(u_j + 2\mathrm{i}\eta)Q(u_j + 2\mathrm{i}\eta)} = -1 \qquad (j = 1, \ldots, K). \tag{5.1}$$

The solutions to these equations which are related to our ODE exhibit various interesting patterns of zeros depending on the choice of (M, N, L, m), and in this section we comment on a few specific examples. First we take one of N, M to be zero. We start with the $M = 0$ case, which is a natural extension of that treated in [19].

When $m = 0$ or 1, the BAE roots assume the famous string patterns of length $L-2$; the number of strings is generically N. We confirmed that this leads to a proper solution of the differential equation (2.1). This means that $\psi_{M,N,m}(x)$ is, modulo a trivial change of variable,

[6] The explicit representation–theoretical construction of the Q operator is not our concern here. However, we mention the recent work [17, 18], where the subtleties which arise for q a root of unity are addressed, taking into account the so-called 'exact complete strings' [24]. It is worth noting that these strings are related to the above-mentioned possibility of multiplying (3.10) by an entire function of x^L.

[7] Takemura [25] has also discussed the application of BAE to the determination of zeros of wavefunctions for the generalized Pöschl–Teller equation (or the BC_1 Calogero–Sutherland model), however, with the L_2 property. The BAE itself is similar to the semi-classical form, thus different from that described here, the quantum form.

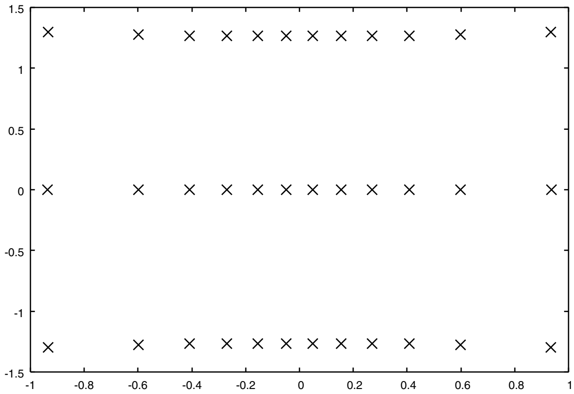

Figure 1. Zeros of Q (2.1) at $(M, N, L, m) = (0, 12, 5, 0)$, which illustrates the three string solution.

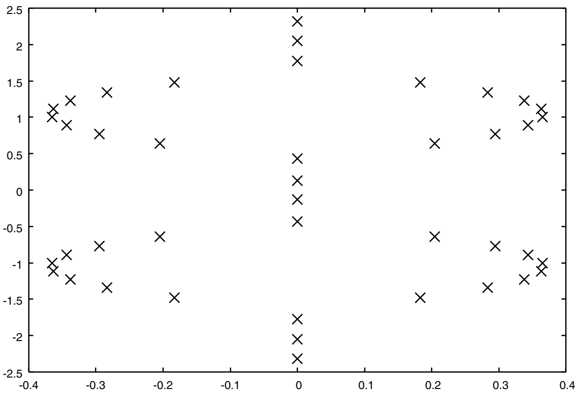

Figure 2. Zeros of the wavefunction of (2.1) with $(M, N, L, m) = (0, 8, 3, 34)$.

Table 1. BAE roots for $L = 4, m = 0, N = 12$.

$\pm 1.170\,531\,980\,09 \pm 0.821\,280\,684\,87$ i	$\pm 0.747\,411\,795\,93 \pm 0.801\,578\,432\,31$ i
$\pm 0.510\,141\,196\,71 \pm 0.796\,245\,853\,90$ i	$\pm 0.336\,827\,053\,94 \pm 0.793\,954\,904\,65$ i
$\pm 0.192\,978\,425\,12 \pm 0.792\,846\,997\,15$ i	$\pm 0.062\,953\,734\,14 \pm 0.792\,380\,188\,77$ i

directly related to the ground-state expectation value of the operator Q. As an example, the set of Bethe ansatz roots for $L = 4$, $m = 0$ and $N = 12$ is given in table 1.

The case with $L = 5$, $m = 0$, $N = 12$ is depicted in figure 1. With increasing m, the roots form longer strings and the number of roots exceeds $(L-2)N$. Finally all but 4N zeros are on the imaginary axis. The remaining 4N zeros lie in four complex groups, which are empirically located near $\pm\epsilon \pm$ i for small real part ϵ. The BAE roots for $(M, N, L, m) = (0, 8, 3, 34)$ are plotted in figure 2. This behaviour will be discussed in appendix A in the light of an explicit solution.

Next we consider the case $M \neq 0$ and $N = 0$. When $m = 0$, $N_k = 0$ and L odd, there are M (almost) strings of length L. The top roots, which are located at $\Im m(u) = \pi$, are displaced from the others: the distance between these roots is slightly larger than others. The example $(M, N, L, m) = (8, 0, 5, 0)$ is shown in figure 3. This configuration can also be interpreted

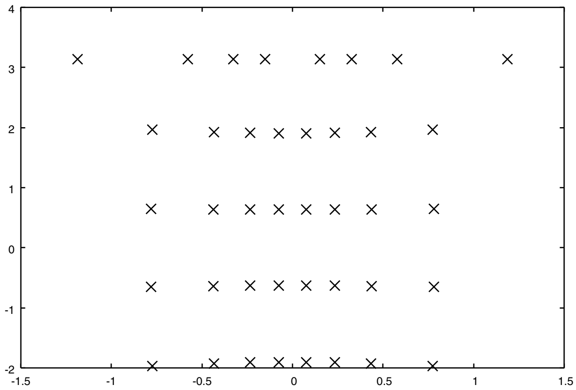

Figure 3. Zeros of the wavefunction of (2.1) with $(M, N, L, m) = (8, 0, 5, 0)$. We have $M-$string in the vertical direction.

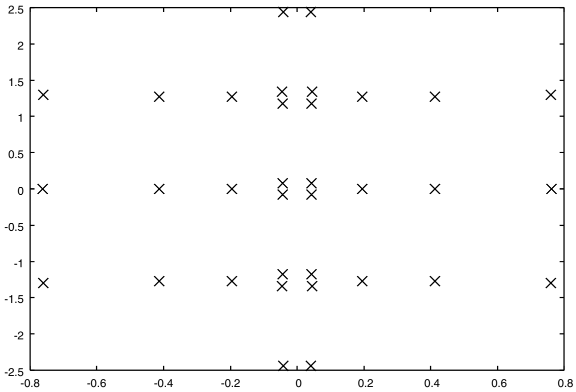

Figure 4. The $(M, N, L, m) = (2, 8, 5, 0)$ case.

as L strings of length M, rotated by 90°. The distance between roots in a string, however, is much less than $\frac{2\pi}{L}$. In this interpretation, the configuration for the case $m = 0$, $N_k = 0$ and L, M even is similar. For $m = 0$, $N_k = 0$, L even and M odd, the pattern is slightly different; there are $L - 2$ strings of length M, a string of length $M - 1$ centred at $x = \pi\mathrm{i}$ and a string of length $M + 1$ centred at $x = 0$.

When both M and N are nonzero, patterns are generally quite involved. However, a very simple picture emerges for $m = 0$: the coexistence of the $L-2$ strings, symmetric with respect to the real axis, and M strings symmetric with respect to the imaginary axis. The example of $L = 5$, $M = 2$, $N = 8$, $m = 0$ is shown in figure 4.

6. The field theory limit

It is interesting to find the field theory limit of our systems. We shall work directly with equation (2.3), and send $N \to \infty$ keeping L, M and m finite. At the same time we focus on the region near $x = 0$ by introducing a new variable z via

$$x^L = \frac{z^L}{L^2 N(N+1)} \tag{6.1}$$

and confine our analysis to the region $|z^L| \ll L^2N(N+1)$. Taking the large N limit, $\psi_{M,N,m}(z) \to \psi_m(z)$ with

$$\frac{\mathrm{d}^2\psi_m(z)}{\mathrm{d}z^2} - \frac{m}{z}\frac{\mathrm{d}\psi_m(z)}{\mathrm{d}z} - z^{L-2}\psi_m(z) = 0. \tag{6.2}$$

Note that at $m = 0$ and $L = 3$, equation (6.2) coincides with the Airy equation (1.4). However, in order to identify the continuous limit of the Q function at $m = 0$, $L = 3$ with the Airy function and hence with result of [1] from the ODE/IM correspondence, we should also check its asymptotic behaviour. (A possible difference in overall normalization will be ignored.) This is the point where the boundary condition (2.5) becomes important. First note that in terms of z, the point where the condition

$$\psi_{M,N,m}(z)|_{z\sim z_0} \sim (z_0 - z)^{N+1} \tag{6.3}$$

was imposed (see equation (2.5)) is $z_0 = (L^2N(N+1))^{1/L}$, so as N is increased z_0 moves towards infinity. At first one might be tempted to extract the large z asymptotic behaviour of $\psi_{M,N,m}(z)$ by studying the large N limit of (6.3). Since we already restricted ourselves to the region $|z^L| \ll L^2N(N+1)$ this would be incorrect: near the point $z = z_0$ ($x = 1$) a linear approximation for the 'potential'

$$P(x) = \left(\frac{L^2N(N+1)x^{L-2}}{(x^L-1)^2} - \frac{L^2M(M+1)x^{L-2}}{(x^L+1)^2}\right) \tag{6.4}$$

is clearly unreasonable. However, it is straightforwardly proved that, for $m = 0$ and $N \geqslant M$, the condition (6.3) constrains $\psi_{N,M,m}(z)$ to be monotonically decreasing in the whole segment

$$0 < z \leqslant (L^2N(N+1))^{1/L}. \tag{6.5}$$

This property guarantees that the purely subdominant solution is singled out from (6.2), giving $q(z) \to \psi_0(z) \propto Ai(z)$. The argument is the following.

The condition (6.3) means that

$$\psi_{M,N,m}(x)|_{x=1-\varepsilon} > 0 \qquad \psi'_{M,N,m}(x)|_{x=1-\varepsilon} < 0 \qquad \psi''_{M,N,m}(x)|_{x=1-\varepsilon} > 0 \tag{6.6}$$

with a small but finite positive ε. So decreasing x slightly below 1, $\psi(x)$ remains positive and in order to change the sign of $\psi'(x)$ the sign of $\psi''(x)$ should become negative first. Note now that for $m = 0$

$$\psi''_{M,N,m}(x) = P(x)\psi_{M,N,m}(x) \tag{6.7}$$

and that $P(x)$ is positive in $0 < x < 1$ for $N \geqslant M$. Then the only way to have $\psi''(x) = 0$ is through $\psi(x) = 0$. By continuity from $x = 1$, this contradicts the positivity condition (6.6).

For $m > 0$, due to the presence of the first derivative term in (2.3), this simple argument does not immediately apply. However, by slightly more involved reasonings one can argue that at least for moderate values of m and M it is always the subdominant solution which is singled out in this field theory limit. For example, the $L = 3$ and $m = 1$ case related to (1.6) of [19] leads to $q(x) \to \psi_1(z) \propto Ai'(x)$.

Finally, we would like to mention that for $m = 0$ and L general the limiting equation (6.2) coincides, up to a trivial change of variable, with the $\alpha = 1$, $l = 0$, $S = (L-2)/2$ case of the equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\chi(x,E)+\left((x^\alpha-E)^{2S}+\frac{l(l+1)}{x^2}\right)\chi(x,E)=0 \tag{6.8}$$

which has been identified by Lukyanov [26] with the scaling limit of the spin$-\frac{L-2}{2}$ XXZ quantum chain.

7. Summary and discussion

In this paper, the generalized Pöschl–Teller (Heun) equation in the complex plane has been addressed in view of the connection relation. Remarkably, at particular values of parameters, a hidden link to one-dimensional quantum systems of higher spins has been found. The Bethe ansatz method, well developed in the theory of quantum integrable systems, then provides a simple characterization of the wavefunction as an entire function. These results begin to fill a gap in earlier studies: they show that the ODE/IM correspondence has a rôle to play in at least some finite lattice systems. The place of massive theories in this story, however, is yet to be clarified.

We would like to remark that a very recent investigation of the deformed nonlinear σ model [27] establishes a connection between perturbed Z_N parafermion theory (a massive theory) and the Heun equation. Though the rôle played by the ODE in the context [27] is quite far from the spirit of the ODE/IM correspondence, it would be nice to see whether the analysis proposed in this paper could tell us anything interesting about the problem [27].

Finally, we add some further comments on the implications of our results for the quantum spin chain problem. The connection rule for the ODE makes full use of the peculiarity of $q = \mathrm{e}^{\mathrm{i}\lambda}$ being a root of unity, which can be naturally extended from $L = 3$ to integer values of L. Correspondingly, some peculiar features of spin model with $S = \frac{1}{2}$ are inherited by spin models for which the quantum space possesses higher spins, while the spin of the auxiliary space remains at $\frac{1}{2}$. Physically, vertex models, or the corresponding Hamiltonians, for which the quantum and the auxiliary spaces share the same magnitude of spin are more relevant. We call these 'isotropic'. Then a natural question arises: can we observe a similarly simple behaviour in the largest eigenvalues of isotropic transfer matrices of higher spin chains, $T \sim (\mathrm{const})^{\mathcal{N}_S}$, with a proper choice of coupling constant? Our preliminary numerical results answer this positively. Under periodic boundary conditions, the 19-vertex model ($L = 4$) and the 44-vertex model ($L = 5$) show the desired simple behaviour for $\mathcal{N}_S = 3, 5, 7$ when $\triangle = -\cos\frac{\pi}{L}$. Indeed, for $L = 5$, this is confirmed by the result in section 4 and fusion relations. Although these eigenvalues are characterized by the same BAE solutions as in the anisotropic cases, for the isotropic models they turn out to be special members of the spectrum—in fact, the largest in the given spin sector. We also investigated Hamiltonians with free–free boundaries. Through numerical diagonalization, the spin 1 chain ($L = 4$) with quantum group invariant boundaries [28] turns out to possess the ground-state energy $E_0 = -4(\mathcal{N}_S - 1)$ for $\triangle = \cos\frac{\pi}{4}$. This is exactly the expected behaviour if T obeys the power law. The origin of this peculiarity, associated with the spin chain, has been argued for $S = \frac{1}{2}$ to be the representation theory of the quantum algebra [29, 30]. There is also an interesting relationship between the antiferromagnetic spin-$\frac{1}{2}$ XXZ quantum chain at $\Delta = -1/2$ and a supersymmetric model of hard-core fermions [31]. It is conceivable that most of the special properties emerging from our analysis will ultimately find a natural interpretation in the framework of similar supersymmetric systems.

We conclude this discussion by noting that the ODE/IM correspondence has been extended in [32–35] to higher order differential equations. In these papers a relationship between nth-order ODEs and the conformal limit of $SU(n)$ lattice models was established. It is interesting that these more complicated families of systems also possess exactly solvable points. (The corresponding differential equations are direct generalizations of the Airy equation (1.4).) At least for these cases, the finite lattice extension of the models should be straightforward, and we hope to explore this point further in a future publication.

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Appendix A. A solution in terms of the Heun function

The ODE (2.1) has an explicit solution in terms of the Heun series $u = H(d, e; \alpha, \beta, \gamma, \delta; z)$ which satisfies

$$\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-d}\right)\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\alpha\beta(z-e)}{z(z-1)(z-d)}u = 0 \tag{A.1}$$

with $\alpha, \beta, \gamma, \delta, \epsilon := \alpha + \beta - \gamma - \delta + 1, d, e \in \mathbf{C}$. (Note that e is denoted by q in [36].) There are four regular singularities at $z = (0, 1, d, \infty)$. It is well known that any Fuchsian function with four regular singular points can be transformed into the Heun function.

Let us make the connection between $\widehat{\psi}$ in (B.1) and the Heun function. Set

$$x^{-\sigma L/2}\widehat{\psi}_{M,N,m}(x) = \frac{\xi}{\tanh^M s \cosh^\sigma s} \qquad (\sigma = (m+1)/L) \tag{A.2}$$

and adopt a variable $z = x^L = -\exp(2s)$. It is then easily established that

$$\xi = (z-1)^\alpha H(d, e; \alpha, \beta, \gamma, \delta; z) \tag{A.3}$$

with parameters

$$\begin{array}{llll} \alpha = \sigma - M - N & \beta = -M - N & \gamma = 1 + \sigma & \delta = -2N \\ \epsilon = -2M & d = -1 & e = \dfrac{M-N}{N+M}. & \end{array} \tag{A.4}$$

The case $M = 0$ and $m \gg 1$ was numerically investigated in the main text. In this limit, it is immediate to see, by its degeneration to the hypergeometric function, that $H_{M,N,m}(x^L) \to (1 - x^L)^N$. Thus

$$\psi_{M,N,m}(x) \to x^{(m+1)/2}(x^{(m+1)/L} - x^{-(m+1)/L}(-1)^N) \tag{A.5}$$

which explicitly supports the asymptotic locations of the zeros being on the unit circle, or on the imaginary axis in terms of u. However, this does not account for the fact that most of them are exactly on the imaginary axis for large but finite m.

Appendix B. The determination of locations of trivial zeros of the wavefunction

In this appendix, we explain how to locate the 'trivial' zeros of the wavefunction. Taking into account the singularities (2.8), there must be special solutions of the form

$$\widehat{\psi}_{M,N,m}(x^L) = \frac{x^{\frac{(2\sigma-M-N)L}{2}}}{(x^{L/2}+x^{-L/2})^M(x^{L/2}-x^{-L/2})^N} H_{M,N,m}(x^L) \qquad \text{(B.1)}$$

where $H_{M,N,m}(z)$ is nonsingular and, as a consequence of (2.10), should satisfy

$$H_{M,N,m}(-z) \propto H_{N,M,m}(z). \qquad \text{(B.2)}$$

As discussed in appendix A, $H_{N,M,m}(z)$ can be written in terms of the Heun function. In view of this explicit form, property (B.2) is immediate: first note that $H_{M,N,m}(x^L) = H(-1, e, \alpha, \beta, \delta; z)$. If $d = -1$, (A.1) is invariant under $z \to -z, e \to -e, \epsilon \leftrightarrow \delta$. This is accomplished in our case by $x^L \to -x^L$, $M \leftrightarrow N$, which can be easily verified using the parametrization (A.4) in terms of σ, M and N. Thus, the desired property (B.2) is shown to be valid.

Using the symmetry $s \to -s$ of the original Pöschl–Teller equation (2.1) one can check that

$$x^{\sigma L}\widehat{\psi}_{M,N,m}(1/x^L) \qquad \text{(B.3)}$$

is also a solution of (2.3), which turns out to be independent of (B.1) as long as $\sigma \notin \mathbf{Z}$. Therefore, the particular solution which satisfies the boundary condition (2.5) is

$$\begin{aligned}\psi_{M,N,m}(x) &= \widehat{\psi}_{M,N,m}(x^L) - (-1)^N x^{\sigma L}\widehat{\psi}_{M,N,m}\left(\frac{1}{x^L}\right)\\ &= \frac{1}{(x^{L/2}+x^{-L/2})^M(x^{L/2}-x^{-L/2})^N}\\ &\quad\times\left(x^{\frac{(2\sigma-M-N)L}{2}}H_{M,N,m}(x^L) - x^{\frac{(M+N)L}{2}}H_{M,N,m}\left(\frac{1}{x^L}\right)\right). \end{aligned} \qquad \text{(B.4)}$$

From the expression (B.4) one can read the positions and orders of the trivial zeros and poles. Consider first zeros related to the symmetry (2.11). We denote the greatest common divisor of $m+1$ and L by

$$\ell = \mathrm{GCD}(m+1, L) \qquad \text{(B.5)}$$

and set

$$\omega = \exp(2\mathrm{i}\eta) \qquad \eta = \frac{\pi}{L} \qquad \text{and} \qquad \omega' = \exp(2\mathrm{i}\eta') \qquad \eta' = \frac{\pi}{\ell}. \qquad \text{(B.6)}$$

Then at $x = \omega'^k$, $(k = 0, 1, \ldots, \ell-1)$, $\psi_{M,N,m}(x)$ has zeros of order $N+1$, and at $x = \omega^k$, $(k = 1, \ldots, L-1)$ such that $\mathrm{GCD}(k\ell, L) = 1$, it has poles of order N.

There are also zeros and poles related to the symmetries (2.10). Let k_i be a positive integer satisfying

$$(2k_i+1)\left(N + M - \frac{m+1}{L}\right) \in 2\mathbf{Z} \qquad (1 \leqslant k_i \leqslant L-1). \qquad \text{(B.7)}$$

By paying attention to (B.4), especially the balance of the two terms in the numerator, we check that $x = \omega^{(2k+1)/2}$ is a pole of the order M of $\psi_{M,N,m}(x)$ if $k \neq k_i$, while it is a zero of the order $M+1$ when $k = k_i$. One can easily check that if k_i is a solution of (B.7) then so also is $k_i + \frac{L}{\ell}$. The number of possible k_i, N_k, is thus either zero or ℓ.

References

[1] Dorey P and Tateo R 1999 Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations *J. Phys. A: Math. Gen.* **32** L419 (*Preprint* hep-th/9812211)

[2] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 2001 Spectral determinants for Schroedinger equation and Q-operators of conformal field theory *J. Stat. Phys.* **102** 567 (*Preprint* hep-th/9812247)

[3] Suzuki J 1999 Anharmonic oscillators, spectral determinant and short exact sequence of $U_q(\widehat{\mathfrak{sl}}(2))$ *J. Phys. A: Math. Gen.* **32** L183 (*Preprint* hep-th/9902053)

[4] Fendley P 1999 Airy functions in the thermodynamic Bethe ansatz *Lett. Math. Phys.* **49** 229 (*Preprint* hep-th/9906114)

[5] Dorey P and Tateo R 1999 On the relation between Stokes multipliers and the T-Q systems of conformal field theory *Nucl. Phys.* B **563** 573
Dorey P and Tateo R 2001 *Nucl. Phys.* B **603** 581 (*Preprint* hep-th/9906219) (erratum)

[6] Suzuki J 2001 Functional relations in Stokes multipliers—Fun with $x^6 + \alpha x^2$ potential *J. Stat. Phys.* **102** 1029 (*Preprint* quant-ph/0003066)

[7] Suzuki J Stokes multipliers, spectral determinants and T-Q relations *Preprint* nlin.si/0009006

[8] Dorey P, Dunning C and Tateo R 2001 Spectral equivalences, Bethe Ansatz equations, and reality properties in PT-symmetric quantum mechanics *J. Phys. A: Math. Gen.* **34** 5679 (*Preprint* hep-th/0103051)

[9] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[10] Sibuya Y 1975 *Global Theory of a Second-Order Linear Ordinary Differential Equation with Polynomial Coefficient* (Amsterdam: North-Holland)

[11] Voros A 1987 Spectral Functions, special functions and Selberg Zeta function *Commun. Math. Phys.* **110** 439

[12] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1996 Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz *Commun. Math. Phys.* **177** 381 (*Preprint* hep-th/9412229)

[13] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 Integrable structure of conformal field theory II. *Q*-operator and DDV equation *Commun. Math. Phys.* **190** 247 (*Preprint* hep-th/9604044)

[14] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1999 Integrable structure of conformal field theory III: The Yang–Baxter relation *Commun. Math. Phys.* **200** 297 (*Preprint* hep-th/9805008)

[15] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 2003 Higher-level eigenvalues of *Q*-operators and Schrödinger equation *Preprint* hep-th/0307108

[16] Rossi M and Weston R 2002 A generalized Q operator for $U_q(\widehat{sl}_2)$ vertex models *J. Phys. A: Math. Gen.* **35** 10015 (*Preprint* math-ph/0207004)

[17] Korff C 2003 Auxiliary matrices for the six vertex model at $q^N = 1$ and a geometric interpretation of its symmetry *J. Phys. A: Math. Gen.* **36** 5229–66 (*Preprint* math-ph/0302002)

[18] Korff C 2003 Auxiliary matrices for the six vertex model at $q^N = 1$ II. Bethe roots, complete strings and the Drinfeld polynomial *Preprint* math-ph/0305035

[19] Fridkin V, Stroganov Y and Zagier D 2000 Ground state of the quantum symmetric finite size XXZ spin chain with anisotropy parameter Delta $= 1/2$ *J. Phys. A: Math. Gen.* **33** L125 (*Preprint* hep-th/9912252)

[20] Stroganov Y 2001 The importance of being odd *J. Phys. A: Math. Gen.* **34** L179 (*Preprint* cond-mat/0012035)

[21] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375–89

[22] Mezincescu L and Nepomechie R I 1992 Analytic Bethe Ansatz for quantum-algebra-invariant spin chains *Nucl. Phys.* B **372** 597–621

[23] Yung C M and Batchelor M T 1995 Integrable vertex and loop models on the square lattice with open boundaries via reflection matrices *Nucl. Phys.* B **435** 430 (*Preprint* hep-th/9410042)

[24] Fabricius K and McCoy B M 2001 Bethe's equation is incomplete for the XXZ model at roots of unity *J. Stat. Phys.* **103** 647–78 (*Preprint* cond-mat/0009279)

[25] Takemura K 2001 The Heun equation and the Calogero-Moser-Sutherland model I: the Bethe ansatz method *Preprint* math.CA/0103077

[26] Lukyanov S L private communication

[27] Fateev V A and Onofri E 2003 An eigenvalue problem related to the nonlinear σ model: analytical and numerical result *Preprint* math-ph/0307010

[28] Pasquier V and Saleur H 1990 Common structures between finite systems and conformal field theories through quantum groups *Nucl. Phys.* B **330** 523

[29] Hinrichsen H, Martin P P, Rittenberg V and Scheunert M 1994 On two point correlation function for the $U_q(SU(2))$ invariant spin one-half Heisenberg chain at roots of unity *Nucl. Phys.* B **330** 533

[30] Martin-Delgado M A and Sierra G 1996 Real space renormalization group method and quantum groups *Phys. Rev. Lett.* **76** 1146

[31] Fendley P, Schoutens K and Nienhuis B 2003 Lattice fermion models with supersymmetry *Preprint* cond-mat/0307338

[32] Dorey P and Tateo R 2000 Differential equations and integrable models: the $SU(3)$ case *Nucl. Phys.* B **571** 583
Dorey P and Tateo R 2001 *Nucl. Phys.* B **603** 582 (*Preprint* hep-th/9910102) (erratum)

[33] Suzuki J 2000 Functional relations in Stokes multipliers and solvable models related to $U_q\left(A_n^{(1)}\right)$ *J. Phys. A: Math. Gen.* **33** 3507 (*Preprint* hep-th/9910215)

[34] Dorey P, Dunning C and Tateo R 2000 Differential equations for general $SU(n)$ Bethe ansatz systems *J. Phys. A: Math. Gen.* **33** 8427 (*Preprint* hep-th/0008039)

[35] Bazhanov V V, Hibberd A N and Khoroshkin S M 2002 Integrable structure of W_3 conformal field theory, quantum Boussinesq theory and boundary affine Toda theory *Nucl. Phys.* B **622** 475 (*Preprint* hep-th/0105177)

[36] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)